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Cylindrical shock and detonation waves in magnetogasdynamics

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Self-similar flow patterns are studied which arise when a cylindrically symmetric strong shock or detonation wave propagates outwards into a gas at rest in which the ambient density varies as the inverse square of the distance from the axis of symmetry along which flows a line current of either zero or finite constant strength. The electrical conductivity of the gas on either side of the wave is supposed perfect and the discontinuities discussed are either gasdynamic or magnetogasdynamic in nature. It is shown that self-similar solutions exist for piston driven gasdynamic detonation and shock waves. Whilst no self-similar solutions may occur for magnetogasdynamic detonation waves, it is demonstrated that magnetogasdynamic shock waves do possess such solutions for which detailed flow patterns are presented.

1. Introduction

The occurrence of self-similar flows in gasdynamics is well established (see, for example, Sedov 1959) and in this paper related problems with discontinuous waves in magnetogasdynamics are studied. The basic configuration investigated is that which arises when an azimuthal magnetic field is generated by a current of finite, constant strength passing along a straight wire of infinite length and either a shock or detonation wave propagates with uniform speed outwards from the wire into the ambient undisturbed gas at rest. The electrical conductivity of the gas on either side of this wave is supposed perfect. The line current may or may not be switched on. The discontinuity is thus either magnetogasdynamic or simply gasdynamic. The model clearly possesses cylindrical symmetry and it is shown that, in the more general case when a detonation wave is propagated, equations of self-similar flow may be established in a certain domain provided that the ambient density varies as the inverse square of the distance from the wire and the ambient pressure ahead of the wave is sufficiently small to be neglected compared with that behind. In order that a comparison may be made between the various types of model this initial distribution of density and pressure is preserved throughout, although it is not essential in all cases. Furthermore, the usual description of a detonation wave is used (see, for example, Helliwell

1963), which in view of the above remarks concerning the pressure must be a strong detonation, consisting of a single discontinuity in the flow at which exothermal energy is released.

A related self-similar problem has been studied by Cole & Greifinger (1962) who investigated the propagation of a magnetogasdynamic cylindrical blast wave into a uniform region of gas permeated by an azimuthal magnetic field generated by a constant line current along the line source of the blast wave. It was found that for a finite energy release the presence of the magnetic field had only a slight influence upon the shock speed, velocity and density distribution, whilst the pressure distribution differed markedly from that in the analogous pure gasdynamic model. These results are in general agreement with those of Greenspan (1962) who carried out similar investigations into the propagation of blast waves across which there occurs a jump of electrical conductivity. We derive similar results for the model appropriate to a magnetogasdynamic shock wave analyzed in detail in the present paper. Greenspan has also remarked that "the existence of a similarity variable and consequent development of a system of ordinary differential equations does not in itself imply the existence of a physically acceptable solution satisfying all the prescribed boundary conditions". This is indeed the case for the present class of problems. We show that no self-similar solution is possible for magnetogasdynamic detonation waves, and only piston driven solutions are possible for gasdynamic detonation waves, all propagating into regions in which the density falls off as the inverse square of the distance from the axis of symmetry.

In a sequel to the study reported here, a subsequent paper will present the results of investigations into the related problems associated with the propagation of ionizing shock and detonation waves for which a jump of electrical conductivity from zero ahead to infinity behind occurs across the wave.

An application of the present work arises in the theory of initiation in explosives by the use of bridge wires. Additional applications may be found in astrophysics in connexion with shock waves in interstellar gas clouds.

2. Self-similar formulation

The fundamental equations governing the continuous flow of a perfect, inviscid gas with infinite electrical conductivity are well established; see, for instance, Ferraro & Plumpton (1961). In the class of the problem under investigation in which either a shock or detonation wave propagates outwards with cylindrical symmetry about a rectilinear current of constant strength I, it is clearly convenient to use cylindrical polar co-ordinates where r is the radial distance from the line current which is taken to pass in a positive sense along the axis of symmetry. Then, using rationalized M.K.S. units, the equations which express the conservation of mass, momentum, energy and magnetic flux in the continuous region behind the wave are, respectively,

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r}(\rho v)+\frac{\rho v}{r}=0, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\mu\mathscr{H}}{\rho r}\frac{\partial}{\partial r}(r\mathscr{H}) = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}(p\rho^{-\gamma})+v\frac{\partial}{\partial r}(p\rho^{-\gamma}) = 0, \tag{2.3}$$

$$\frac{\partial \mathscr{H}}{\partial t}+\frac{\partial}{\partial r}(v\mathscr{H}) = 0. \tag{2.4}$$

Here, p is the pressure, ρ the density, v the radial velocity, $\mathscr{H}$ the azimuthal magnetic field, γ the adiabatic gas index, μ the magnetic permeability, t the time, and r is defined above.

The basic jump conditions across a discontinuous wave in magnetogasdynamics have been given in general by Helliwell & Pack (1962) without restriction upon the electrical conductivity on either side of the wave. With the quantities referred to the fixed cylindrical co-ordinate system, and γ, μ supposed constant throughout, these are

$$\rho_2(c-v_2) + \rho_1 c, \tag{2.5}$$

$$\rho_2(c-v_2)^2+p_2+\tfrac{1}{2}\mu\mathscr{H}_2^2 = \rho_1 c^2+p_1+\tfrac{1}{2}\mu\mathscr{H}_1^2, \tag{2.6}$$

$$\tfrac{1}{2}(c-v_2)^2+\frac{\gamma p_2}{(\gamma-1)\rho_2}+\frac{E_2\mathscr{H}_2+\mu c\mathscr{H}_2^2}{\rho_2(c-v_2)} = \tfrac{1}{2}c^2+\frac{\gamma p_1}{(\gamma-1)\rho_1}+\frac{E_1\mathscr{H}_1+\mu c\mathscr{H}_1^2}{\rho_1 c}+\bar{Q}, \tag{2.7}$$

$$E_2+\mu c\mathscr{H}_2 = E_1+\mu c\mathscr{H}_1, \tag{2.8}$$

where subscripts 1 and 2 refer to upstream and downstream conditions respectively, E is the electric field, c is the wave speed and the gas is supposed at rest upstream. In the case of a shock wave $\bar{Q} = 0$, but for a detonation wave $\bar{Q}$ is the exothermal energy per unit mass released at the wave front. In addition to this set of equations must be added further relationships whose form depends essentially upon the assumption that the gas everywhere has infinite conductivity. Thus in order that infinitely large currents do not develop, it is required from Ohm's law that, downstream,

$$E_2+\mu v_2\mathscr{H}_2 = 0, \tag{2.9}$$

whilst upstream a similar argument leads to the requirement that

$$E_1 = 0. \tag{2.10}$$

The formulation of the problem is completed by the specification of the conditions upstream of the discontinuous wave. It has already been assumed that the gas in this region is at rest, so that the remaining quantities which call for consideration are p_1, ρ_1 and $\mathscr{H}_1$, where in view of the axial current, I, one has

$$\mathscr{H}_1 = I/2\pi r. \tag{2.11}$$

Now if it is to be possible for a self-similar solution to exist then, of the set of dimensional parameters which determine the flow, only two may be dimensionally independent. In the present class of problem, let us suppose that the upstream density varies inversely as some power of the distance from the axial current, so that

$$\rho_1 = \rho_0 r^{-\omega}. \tag{2.12}$$

Then it will be observed that, since γ is dimensionless and μ always arises combined with $\mathscr{H}^2$ multiplicatively, the basic dimensional parameters are p_1, ρ_0, μI^2 and $\bar{Q}$ together with the wave speed c. Further since $[\mu\mathscr{H}^2] = [p]$, it follows that

$$[p_1] = ML^{-1}T^{-2}, \quad [\rho_0] = ML^{\omega-3},$$
$$[\mu I^2] = MLT^{-2}, \quad [Q] = L^2T^{-2}, \quad [c] = LT^{-1}.$$

Thus, in the general case when $\bar{Q}$ and I are both non-zero a self-similar flow pattern may be possible only when $\omega = 2$ and $p_1 = 0$. It should be noted that a non-uniform upstream pressure is unable to be supported as it would infringe the momentum equation (2.2) in the region ahead of the wave. Further the neglect of the upstream pressure is equivalent to the supposition that the quantity $(p_1 r^2/c^2\rho_0)$ is both small and uninfluential, an assumption which is clearly not valid for all space. The solution is consequently restricted to a domain $r < (c^2\rho_0/p_1)^{\frac{1}{2}}$, a restriction which applies equally to the earlier work of Cole & Greifinger and Greenspan. Assuming this condition satisfied we restrict further consideration to the situation when the ambient density varies as the inverse square of the distance from the axial current and the discontinuous wave is sufficiently strong for the upstream pressure to be neglected compared with that downstream. It follows therefore that the two independent dimensional parameters A and B, which arise, may be taken with the dimensions

$$[A] = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}, \quad [B] = LT^{-1}, \tag{2.13}$$

and a similarity variable, λ, may be defined by

$$\lambda = \frac{\beta r}{Bt}, \tag{2.14}$$

where β is a dimensionless constant. Then the velocity, pressure, density and magnetic field may be written in terms of corresponding non-dimensional functions of λ. Thus

$$v = \frac{r}{t}V(\lambda), \quad p = \frac{A^2}{r^2}P(\lambda), \quad \rho = \frac{A^2t^2}{r^4}R(\lambda), \quad \sqrt{(\mu\mathscr{H})} = \frac{A}{r}H(\lambda). \tag{2.15}$$

Finally, let us suppose that the position co-ordinate of the discontinuous wave front is $r = r^*$, at which $\lambda = \lambda^*$. Then β may be chosen so that $\lambda^* = 1$, and from (2.14) we have

$$r^* = Bt/\beta, \tag{2.16}$$

so that

$$c = \frac{dr^*}{dt} = \frac{B}{\beta} = \frac{r^*}{t}. \tag{2.17}$$

Hence the wave speed is constant for the class of self-similar solution under investigation.

In the case of detonation waves it is convenient to choose

$$A = \rho_0^{\frac{1}{2}}c, \quad B = \bar{Q}^{\frac{1}{2}}, \tag{2.18}$$

so that, from equations (2.14) and (2.17), the similarity variable is given by

$$\lambda = \frac{\beta r}{t\bar{Q}^{\frac{1}{2}}}, \quad \beta = \frac{\bar{Q}^{\frac{1}{2}}}{c}. \tag{2.19}$$

However, in the case of shock waves for which $\bar{Q} = 0$, one must choose instead

$$A = \rho_0^{\frac{1}{2}} c, \quad B = c, \tag{2.20}$$

so that, since from (2.17) it follows that $\beta = 1$, one has

$$\lambda = \frac{r}{ct}. \tag{2.21}$$

The equations governing the flow may now be written in non-dimensional form.

The use of the transformation (2.14) and substitution from (2.15) into (2.1)–(2.4) leads to a set of ordinary differential equations, governing the continuous flow behind the discontinuous wave, which may be written

$$\lambda\left\{-V' + \frac{(1-V)R'}{R}\right\} = 2(1-V), \tag{2.22}$$

$$\lambda\left\{(1-V)V' - \frac{P'}{R} - \frac{HH'}{R}\right\} = -\frac{2P}{R} - V(1-V), \tag{2.23}$$

$$\lambda(1-V)\left\{\frac{P'}{P} - \frac{\gamma R'}{R}\right\} = -2\gamma + 2(2\gamma-1)V, \tag{2.24}$$

$$(VH)' - H' = 0, \tag{2.25}$$

where the prime denotes differentiation with respect to λ. The equations are valid in the inteval $0 \leqslant \lambda < 1$.

In a similar manner, from the equations (2.5)–(2.12) one may derive the dimensionless form of the jump relations across the shock or detonation wave. These relate the values of H, P, R and V at $\lambda = 1$, upstream and downstream, of which the latter provide a set of boundary conditions for (2.22)–(2.25). We find, upstream,

$$H = \frac{\sqrt{(\mu)}I}{2\pi c\sqrt{\rho_0}}, \quad P_1 = 0, \quad R_1 = 1, \quad V_1 = 0. \tag{2.26}$$

Immediately downstream of the discontinuous wave it follows that

$$R_2(1-V_2) = 1, \tag{2.27}$$

$$R_2(1-V_2)^2 + P_2 + \tfrac{1}{2}H_2^2 = 1 + \tfrac{1}{2}H_1^2, \tag{2.28}$$

$$\tfrac{1}{2}(1-V_2)^2 + \frac{\gamma P_2}{(\gamma-1)R_2} + \frac{H_2^2}{R_2} = \tfrac{1}{2} + H_1^2 + Q, \quad \left(Q = \frac{\bar{Q}}{c^2}\right), \tag{2.29}$$

$$H_2(1-V_2) = H_1. \tag{2.30}$$

In addition, it should be noted that the equations for a purely gasdynamic problem in the absence of the axial current are derived from the system (2.22)–(2.30) by setting $H = H_1 = H_2 \equiv 0$ throughout.

Before leaving this section it is useful to establish two further results which will be found of importance in the further development of solutions. At a piston face the piston speed is identical with that of fluid particles in contact, so that

$$V = 1. \tag{2.31}$$

Furthermore, the speed of propagation of a magneto-acoustic wave front is given by the slope of the characteristics in the (r, t) plane of the fundamental equations (2.1)–(2.4). Hence, after substitution from (2.15) one finds that a magneto-acoustic wave front occurs in the gas where

$$R(1-V)^2 = \gamma P + H^2. \tag{2.32}$$

3. The equations of continuous flow

The equations of continuous flow comprise the set (2.22)–(2.25). Three integrals, corresponding to conservation of mass, magnetic flux and energy between two similarity surfaces λ = constant, may be obtained as described by Sedov (1959), or, as here, by simple manipulation of the governing equations.

Equation (2.25) integrates directly to yield

$$H(1-V) = \text{constant} = a. \tag{3.1}$$

This expresses conservation of magnetic flux.

Next, after division by $\lambda(1-V)$, (2.22) becomes separable and integration followed by an application of the condition (2.27) at $\lambda = 1$ leads to the mass integral in the form

$$R(1-V) = \lambda^2. \tag{3.2}$$

The energy integral may be obtained similarly. First, add $\frac{1}{2}RV^2$ times (2.22) and RV times (2.23). Next derive the sum of γ times (2.22) and (2.24). Then, by addition, one obtains an equation which by means of (2.25) may be shown to be exact and to possess an integral which by the use of (3.1) and (3.2) may be written

$$P = \frac{\gamma-1}{2(\gamma V-1)(1-V)^2}\{2b(1-V)^2 + V^2[\lambda^2(1-V)^2 - a^2]\}. \tag{3.3}$$

A single ordinary differential equation for $V = V(\lambda)$ may now be obtained from the addition of (2.23) and (2.24) multiplied respectively by $R(1-V)$ and P. One derives, following further application of (3.1) and (3.2) the governing equation

$$\frac{d\lambda^2}{dV} = \frac{\lambda^2\{2(\gamma V-1)[\lambda^2(1-V)^3 - a^2] - \gamma(\gamma-1)[2b(1-V)^2 + V^2\{\lambda^2(1-V)^2 - a^2\}]\}}{(\gamma V-1)\{(\gamma-1)[2b(1-V)^2 + V^2\{\lambda^2(1-V)^2 - a^2\}] - \lambda^2 V(1-V)^3\}}. \tag{3.4}$$

Clearly the solution is dependent upon the values of the constants a and b, which are themselves dependent upon the type of discontinuous wave which is studied. Qualitative information concerning possible flow patterns may be obtained from a study of the integral curves of (3.4). In what follows it will be found that these curves are very sensitive to the values of a and b which for the class of problem under investigation are such that $a \geqslant 0$, $b \leqslant 0$.

Finally it should be noted that the locus of magneto-acoustic wave fronts in the (λ^2, V) plane is given from (2.32) by eliminating R, P and H through the use of (3.1)–(3.3). One finds

$$2(\gamma V-1)[\lambda^2(1-V)^3 - a^2] - \gamma(\gamma-1)[2b(1-V)^2 + V^2\{\lambda^2(1-V)^2 - a^2\}] = 0, \tag{3.5}$$

which is clearly a locus of extrema on the integral curves of the governing equation (3.4).

4. The shock relations

Consider the jump relations across the magnetogasdynamic wave as given by (2.27)–(2.30). From the first of these, since $R_2 \geqslant 0$, it follows that $0 \leqslant V_2 \leqslant 1$. The elimination of H_2, P_2 and R_2 between the full set of equations then leads to a cubic equation for V_2, which may be written in the form

$$F(V, H_1, Q) \equiv (\gamma+1)V^3-(\gamma+3)V^2+2V+\{(\gamma V-2)V\}H_1^2+2(\gamma-1)(V-1)Q = 0, \tag{4.1}$$

where the subscript 2 on V_2 has been omitted.

Now, since for all non-zero (H_1, Q) the function F is positive or negative according as $V \to \pm\infty$ respectively, whilst $F < 0$ at both $V = 0$ and $V = 1$, it follows that the cubic equation has at least one real root greater than unity and either two real roots or none less than zero. It should also be noted that both $F'(1, H_1, Q)$ and $F'(0, H_1, Q)$ are positive if $H_1 < 1$, where the prime denotes differentiation with respect to V. Thus, since we have already observed that to be physically realistic the roots must lie in (0, 1), it follows that the cubic may have no negative roots. Further, from (2.27) and (2.28) one finds

$$\{P_2+\tfrac{1}{2}H_2^2\}-V_2 = \{\tfrac{1}{2}H_1^2\},$$

where the quantities in braces denote the total pressure on the downstream and upstream sides of the discontinuity. Therefore since, as has already been remarked, the shock or detonation wave must be strong, one must choose the larger of the two roots in the range (0, 1). In this range $F(V, H_1, Q)$ is a decreasing function of both H_1 and Q, that is

$$\left.\begin{matrix} F(V,0,0) > F(V,H_1,0) \\ F(V,0,0) > F(V,0,Q) \end{matrix}\right\} > F(V,H_1,Q). \tag{4.2}$$

It is easily shown that if the cubic $F(V, 0, Q) = 0$ has real roots in $0 \leqslant V \leqslant 1$, then

$$Q < \frac{1}{2(\gamma^2-1)}, \tag{4.3}$$

whilst the cubic $F(V, H_1, 0) = 0$ has real roots in the same range if

$$H_1 < 1. \tag{4.4}$$

Note that this last condition implies that the shock speed is greater than the Alfvén speed at the wave front. Then in view of the relations (4.2) it follows that for physically realistic roots of the cubic equation (4.1) to occur, the inequalities (4.3) and (4.4) give rough upper bounds for Q and H_1 respectively.

Suppose that the turning points of the cubic equation (4.1) viz., the roots of the quadratic equation

$$F'(V, H_1, Q) = 0 \tag{4.5}$$

are α and β. Then

$$\alpha+\beta = \frac{2(3+\gamma)-\gamma H_1^2}{3(\gamma+1)}, \quad \alpha\beta = \frac{2\{1-H_1^2+Q(\gamma-1)\}}{3(\gamma+1)},$$

both of which are positive, since $H_1^2 < 1$. Hence both turning points of the cubic equation (4.1) lie in $V \geqslant 0$, and further, since $(\alpha+\beta) < 2$ at least one of these is in the range $(0, 1)$. However, $F'(1, H, Q) > 0$, so that both turning points must lie in $(0, 1)$. Thus, of the real roots of the cubic equation (4.1) one always exists greater than unity whilst two others arise in the range $(0, 1)$ provided that

$$F(\alpha, H_1, Q) . F(\beta, H_1, Q) \leqslant 0, \tag{4.6}$$

where the equality is restricted to the case of two coincident roots. After considerable algebra this condition may be written in the form

$$\begin{aligned}
&8(\gamma^2-1)(\gamma-1)^2Q^3-(\gamma-1)^2\{\gamma^2H_1^4-4(5\gamma^2-6)H_1^2-4(2\gamma^2-3)\}Q^2\\
&\quad -2(\gamma-1)\{\gamma^2(\gamma-1)H_1^6-3(\gamma^3-\gamma^2-\gamma+4)H_1^4+3(\gamma^3-\gamma^2-2\gamma-4)H_1^2\\
&\quad -(\gamma-1)(\gamma^2-3)\}Q-\{\gamma^2H_1^8-2(2\gamma^2-\gamma-4)H_1^6+3(2\gamma^2-2\gamma-5)H_1^4\\
&\quad -2(2\gamma^2-3\gamma-3)H_1^2+(\gamma-1)^2\} \leqslant 0.
\end{aligned} \tag{4.7}$$

By setting either $H_1 = 0$ or $Q = 0$ the conditions (4.3) and (4.4) are recovered as special cases of the general result. Provided that values of H_1 and Q are chosen which satisfy the inequality (4.7), (4.1) has three real roots the middle one of which gives an acceptable solution for V_2 appropriate to strong shock or detonation waves. It then follows from (2.27)–(2.30) that

$$R_2 = \frac{1}{1-V_2}, \tag{4.8}$$

$$H_2 = \frac{H_1}{1-V_2}, \tag{4.9}$$

$$P_2 = \frac{V_2\{2(1-V_2)^2-H_1^2(2-V_2)\}}{2(1-V_2)^2}. \tag{4.10}$$

The constants a and b which arise in the integrals of the equations of continuous flow may now be determined. The constant a is defined in (3.1) by

$$a = H_2(1-V_2),$$

which by the use of (4.9) becomes

$$a = H_1. \tag{4.11}$$

Also from the insertion of the expression for P_2 from (4.10) into the integral (3.3), since V_2 satisfies $F(V, H_1, Q) = 0$, one obtains, after simplification, the result

$$b = -Q. \tag{4.12}$$

The case of a gasdynamic wave is obtained as a special case of the above results by setting $H_1 = 0$. In particular one finds that for a strong discontinuity V_2 must be taken as the middle root of the appropriately modified equation (4.1), where Q is subject to the condition (4.3). Hence

$$V_2 = \frac{1+\{1-2Q(\gamma^2-1)\}^{\frac{1}{2}}}{\gamma+1}, \tag{4.13}$$

so that, since Q is bounded, $$\frac{1}{\gamma+1} \leqslant V_2 \leqslant \frac{2}{\gamma+1}. \tag{4.14}$$

The various classes of wave and associated values of (a, b) may thus be listed as follows: (i) $a = 0$, $b < 0$: gasdynamic detonation wave. (ii) $a = 0$, $b = 0$: gasdynamic shock wave. (iii) $a > 0, b < 0$: magnetogasdynamic detonation wave. (iv) $a > 0$, $b = 0$: magnetogasdynamic shock wave.

5. Gasdynamic detonation and shock waves

Let us first consider the case of a pure gasdynamic detonation wave for which $a = 0$ and $b = -Q$. If one puts

$$y = \lambda^2/Q, \quad x = V,$$

the governing equation (3.4) becomes

$$\frac{dy}{dx} = \frac{y\{2(\gamma x-1)(1-x)y-\gamma(\gamma-1)(x^2y-2)\}}{(\gamma x-1)\{(\gamma-1)(x^2y-2)-xy(1-x)\}}. \tag{5.1}$$

This has singularities in the finite part of the (x, y) plane at the points

$$\left.\begin{array}{ll} (a)\; x = 1/\gamma,\, y = 0; & (b)\; x = 1/\gamma,\, y = 2\gamma^2; \\ (c)\; x = 1, \quad y = 2; & (d)\; x = 2/\gamma,\, y = \gamma(\gamma-1). \end{array}\right\} \tag{5.2}$$

It is a straightforward matter to show that these four singularities are all simple and have the following characters:

(*a*) saddle, (*b*) singular node, (*c*) saddle, (*d*) spiral.

The numerator of equation (5.1) is zero on $y = 0$ and

$$y = \frac{2\gamma(\gamma-1)}{\gamma(\gamma+1)x^2-2(\gamma+1)x+2} \tag{5.3}$$

which, on comparison with equations (3.8) is seen to be the locus of acoustic wave fronts. For $\gamma > 1$ the denominator of (5.3) is a positive definite form with minimum at $x = 1/\gamma$. Further, the denominator of equation (5.1) is zero on $x = 1/\gamma$ and

$$y = \frac{2(\gamma-1)}{(\gamma x-1)x}, \tag{5.4}$$

which is positive for $0 < x < 1/\gamma$. Hence, by the use of these results, one may obtain an accurate representation of the pattern of integral curves for equation (5.1). This is shown in figure 1.

On these integral curves the initial point corresponding to conditions immediately behind the detonation wave at $\lambda = 0$ is $x = V_2 = V_2(Q)$, $y = 1/Q$. This point lies above the singularity (*b*) if $Q < 1/(2\gamma^2)$ so that from (4.13) and inequalities (4.14) one finds $1/\gamma < x \leqslant 2/(\gamma+1)$. Similarly, if this point lies below the singularity (*b*) then $1/(\gamma+1) \leqslant x < 1/\gamma$. Hence the initial point lies, within the ranges specified, above and to the right or below and to the left of singularity (*b*), and a solution is obtained by passage along an integral curve in the direction of decreasing λ. The locus of these initial points for varying Q may be expressed analytically in the form

$$xy[(\gamma+1)x-2]+2(\gamma-1) = 0. \tag{5.5}$$

This shock locus is shown in figure 1 superimposed upon the pattern of integral curves.

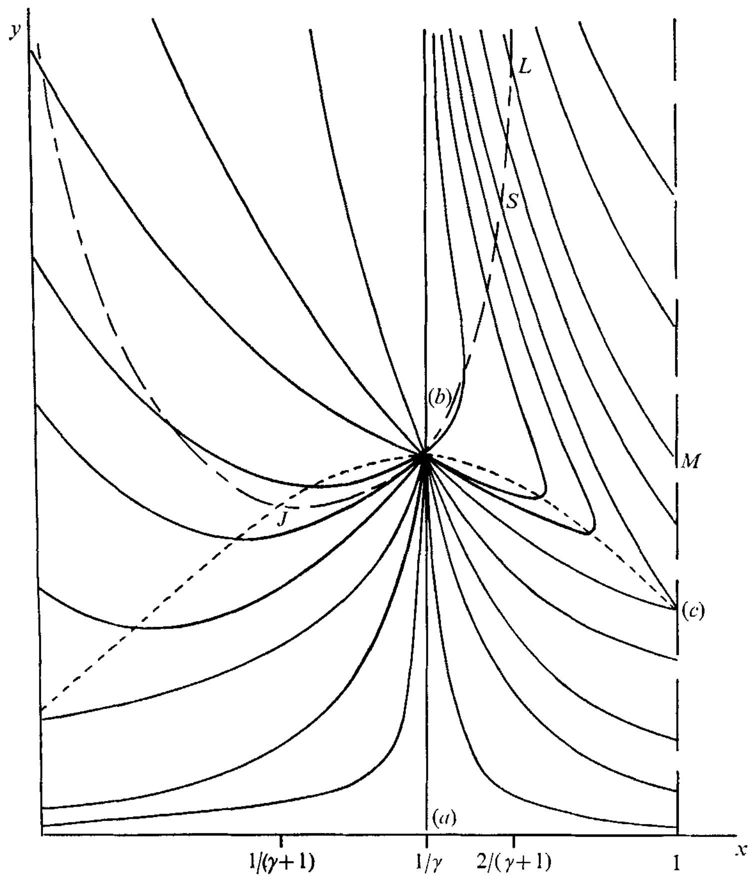

FIGURE 1. Gasdynamic detonation wave: integral curves. – – –, acoustic locus; —— – ——, shock locus.

Now every integral curve starting from a point below and to the left of (*b*) intersects the acoustic curve at which the sense of variation of λ changes. Consequently there is no solution in such cases. An integral curve starting from a point above and to the right of (*b*) may intersect the acoustic curve, which, as before, gives no solution, or it may intersect the line $V = 1$, which corresponds to a piston path. Such integral curves would thus be associated with piston driven detonation waves. A further possibility is for the integral curve to enter and pass through the singularity (*b*) in which case the solution would have a weak singularity at that point, and the velocity, density and pressure would be continuous. However, whilst the velocity and density are unique at (*b*), the pressure is given from (3.3) by

$$P = \frac{Q(\gamma-1)(x^2y-2)}{2(\gamma x-1)}, \tag{5.6}$$

which is undetermined there; its value depends upon the slope of the particular integral curve at entry into (*b*). Indeed, if this slope be m, then

$$P = \frac{Q(\gamma-1)(4\gamma^3+m)}{2\gamma^3}.$$

Therefore any integral curve which starts above and to the right of (*b*) and enters (*b*) before crossing the acoustic curve, necessarily passes subsequently below and to the left of (*b*) and then crosses the acoustic curve thereby failing to yield an acceptable solution. The final possibility is for the integral curve to start from (*b*) itself. However, in this case immediately behind the detonation wave one finds from (4.10) that $P_2 = V_2$ so that from (5.7) and inequality (4.3) it follows that $m > 0$. Thus the integral curve must pass below and to the left of (*b*) and hence fail to provide a solution.

It is of interest to note that there exists no self-similar solution for the Chapman–Jouguet detonation wave, for the Chapman–Jouguet point lies at the intersection of the shock locus and acoustic locus, where $V_2 = 1/(\gamma+1)$. This point is shown labelled J on figure 1 and lies below and to the left of the singularity (*b*).

From the above discussion one is led to the conclusion that only piston-driven detonation waves may have self-similar character in a pure gasdynamic model with an inverse square law of variation of ambient density. Acceptable solutions are given by integral curves of the type LM indicated in figure 1 which stem from points on the shock locus above S, the point of its intersection with the unique integral curve which enters the saddle point (*c*) and does not pass through the singularity (*b*). The location of S may be established numerically by integration from the singularity (*c*) along the limiting integral curve until the shock locus is intercepted. The outcome of such a calculation is to establish the range of values of Q for which self-similar solutions exist. For example, if $\gamma = \frac{5}{3}$ one finds that piston driven waves exist provided that $0 < Q \leqslant 0{\cdot}0438$, correct to 3 significant figures. For given Q the details of the flow pattern between the piston and the detonation wave may now be established by a straightforward numerical integration of (5.1) from the shock point $x = 1/Q$, $y = V_2(Q)$ to the piston point $x = 1$. The solution is most usefully presented in terms of the ratios of the values of the field variables at a general point to their values immediately behind the detonation wave, expressed as functions of the proportional distance r/r^* from the current axis to the wave front. In terms of x, y and Q these are given by

$$\frac{v}{v^*} = \frac{x}{V_2}(Qy)^{\frac{1}{2}}, \quad \frac{\rho}{\rho^*} = \frac{1-V_2}{Qy(1-x)}, \quad \frac{p}{p^*} = \frac{(\gamma-1)(x^2y-2)}{2(\gamma x-1)yV_2}, \quad \frac{r}{r^*} = (Qy)^{\frac{1}{2}}, \tag{5.7}$$

where $V_2 = V_2(Q)$ is expressed in (4.13), and, as before an asterisk denotes values at the wave front. Furthermore, the piston speed, c_p, is given in terms of the

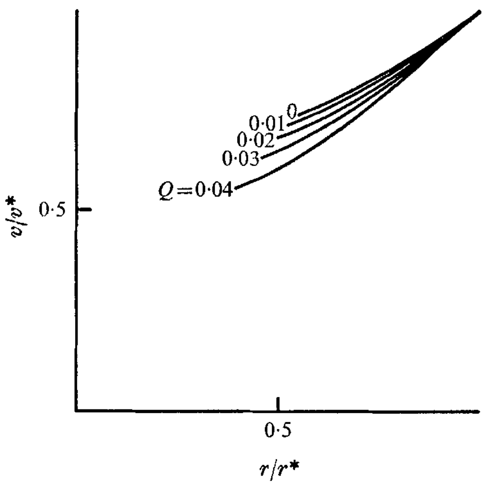

FIGURE 2 (a). Gasdynamic detonation wave : velocity distribution.

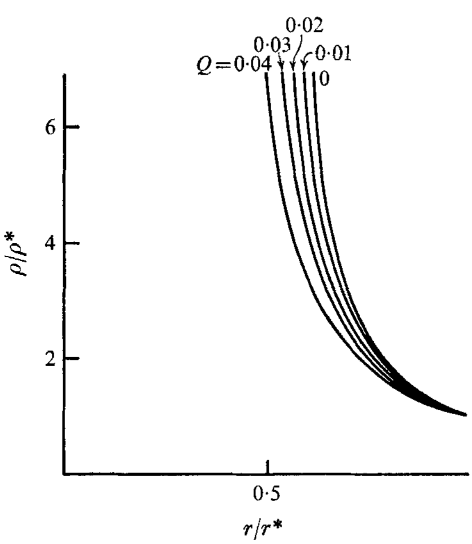

FIGURE 2 (b). Gasdynamic detonation wave : density distribution.

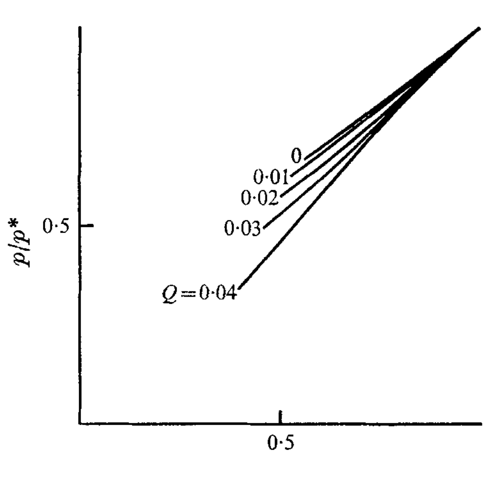

FIGURE 2 (c). Gasdynamic detonation wave : pressure distribution.

detonation speed, c, by $c_p = (Qy_p)^{\frac{1}{2}}c$, where y_p is the value of y on the solution curve where $x = 1$. Complete solutions have been calculated, correct to 3 significant figures, for $\gamma = \frac{5}{3}$ and the values of Q (except $Q = 0$ and $Q = 0{\cdot}0438$) listed, together with corresponding values of c_p/c in table 1. The results are portrayed graphically in figures 2(*a*)–(*c*).

Q	c_p/c	Q	c_p/c
0	0·554	0·03	0·458
0·01	0·527	0·04	0·394
0·02	0·497	0·0438	0·296

TABLE 1

The alternative gasdynamic model which may be studied is that of a piston-driven gasdynamic shock wave with no exothermal energy release at the wave front. Thus, with $Q = 0$, we define $\bar{x} = V$ and $\bar{y} = \lambda^2$ so that the governing equation (3.4) then becomes

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}\{(\gamma+1)\bar{x}(2-\gamma\bar{x})-2\}}{\bar{x}(\gamma\bar{x}-1)^2}, \tag{5.8}$$

which, under the boundary conditions $\bar{x} = 2/(\gamma+1), \bar{y} = 1$, has the exact solution

$$\bar{y}\bar{x}^2(\gamma+1)^2\left[\frac{\gamma-1}{\gamma+1}\right]^{(\gamma-1)/\gamma}\exp\left[\frac{\gamma+1}{\gamma}\right] = 4(\gamma\bar{x}-1)^{(\gamma-1)/\gamma}\exp\left\{\frac{\gamma-1}{\gamma(\gamma\bar{x}-1)}\right\}. \tag{5.9}$$

The section of the solution curve between $\bar{x} = 2/(\gamma+1)$ and $\bar{x} = 1$ is associated with the flow patterns of piston driven shock waves such that, since $c_p/c = \bar{y}_p^{\frac{1}{2}}$ where $\bar{y}_p = (\bar{y})_{x=1}$,

$$\frac{c_p}{c} = 2(\gamma+1)^{-(\gamma+1)/(2\gamma)}\exp\left(-\tfrac{1}{2}\right).$$

For $\gamma = \frac{5}{3}$ one finds $c_p/c = 0{\cdot}554$ which, for completeness has been included in table 1. In a similar manner to that for the detonation wave the flow pattern may be presented functionally in terms of $\bar{x}$, $\bar{y}$ as follows

$$\frac{v}{v^*} = \frac{(\gamma+1)\bar{x}\bar{y}}{2}, \quad \frac{\rho}{\rho^*} = \frac{\gamma-1}{(\gamma+1)\bar{y}(1-\bar{x})}, \quad \frac{p}{p^*} = \frac{(\gamma^2-1)\bar{x}^2}{4(\gamma\bar{x}-1)}, \quad \frac{r}{r^*} = \bar{y}^{-\frac{1}{2}}.$$

The distributions of these quantities in the region between the piston and the shock wave are also shown in figure 2.

For both shock and detonation waves, at a fixed value of r/r^* all the quantities v/v^*, ρ/ρ^*, p/p^* decrease as Q increases, whilst it is easy to show that the temperature ratio increases. At the piston face whilst the velocity and pressure are in general finite non-zero, there the density becomes infinitely large and, in consequence, the temperature zero. This singular behaviour is not unexpected, since, in the ambient state $\rho \to \infty$ as $r \to 0$, which is the location of the piston face at some initial instant. Furthermore, the velocity ratio and pressure ratio are fairly sensitive to the value of Q. As has been already noted, to the maximum value of Q corresponds the integral curve (S)–(c) of figure 1. Now from the expression (5.6) it follows that the pressure is zero on $y = 2/x^2$ which curve passes through both singularities (b) and (c) and may also be shown to satisfy the governing equation (5.1). Hence, for $1/\gamma < x < 1$, the pressure falls to zero only at points lying on the integral curve (b)–(c), being positive at points lying above. Consequently all solutions have non-zero pressure throughout, apart from that corresponding to this maximum value of Q, for which the pressure falls to zero at the piston. In all cases, immediately behind the wave front the pressure decreases rapidly. This is in general agreement with a statement made by Stanyukovich (1960) regarding spherical and cylindrical detonation waves propagating into uniform media.

6. Magnetogasdynamic detonation waves

In the case of a magnetogasdynamic detonation wave it has been seen that $a = H$ and $b = -Q$. Thus the governing equation (3.4) for the flow behind this wave becomes

$$\frac{dy}{dx} = \frac{y\{2(\gamma x-1)[y(1-x)^3-h]-\gamma(\gamma-1)[(1-x)^2(x^2y-2)-x^2h]\}}{(\gamma x-1)\{(\gamma-1)[(1-x)^2(x^2y-2)-x^2h]-xy(1-x)^3\}}, \tag{6.1}$$

where
$$x = V, \quad y = \lambda^2/Q, \quad h = H_1^2/Q. \tag{6.2}$$

Clearly the parameter h can take all values between zero and infinity, although here we shall exclude the special cases $Q = 0$ and $h = 0$, the latter of which has been studied as a special gasdynamic case in § 5.

Two singularities of equation (6.1) lie at the points

$$\left.\begin{array}{ll}(a) & x = 1/\gamma, \quad y = 0;\\ (b) & x = 1/\gamma, \quad y = \gamma^2\{2(\gamma-1)^2+h\}/(\gamma-1)^2.\end{array}\right\} \tag{6.3}$$

Their nature is found to be independent of h; they correspond to a saddle point and a singluar node respectively. Further singularities are given by the points of intersection of the curves represented by the numerator and denominator of (6.1) equated separately to zero, viz.

$$2(\gamma x-1)[y(1-x)^3-h]-\gamma(\gamma-1)[(1-x)^2(x^2y-2)-x^2h] = 0, \tag{6.4}$$

$$(\gamma-1)[(1-x)^2(x^2y-2)-x^2h]-xy(1-x)^3 = 0, \tag{6.5}$$

the first of these being the magneto-acoustic locus. It may be shown that there are no solutions of the simultaneous equations (6.4), (6.5) arising in the range $0 \leqslant x \leqslant 1$. Consequently the only singularity of (6.1) in this range are those already defined in (6.3).

From a study of the integral curves it is apparent, for all h, that as one passes along an integral curve in the sense of decreasing λ starting from a point with $x \neq 1/\gamma$ in $0 < x < 1$ one must either pass through the singular point (b) and intersect the magneto-acoustic curve or intersect the latter directly. Neither of these possibilities offers a solution. Further on the integral curve $x = 1/\gamma$ it is clear from (3.6) that P is infinite except at the singularity (b), so that no physically meaningful solution is offered in this case. Consequently one must conclude that no self-similar solution is possible for this problem; there is no possibility of a self-sustained Chapman–Jouguet detonation, nor of a detonation wave driven by a piston with uniform speed. Of course a solution may exist for a detonation wave of non-uniform speed driven by a variable speed piston, but such a solution would not be self similar and is beyond the scope of this investigation.

7. Magnetogasdynamic shock waves

In the case of a shock wave which propagates into a gas with perfect electrical conductivity one has $b = 0$ and $a = H_1$. Thus, if in equation (3.4) one sets

$$x = V, \quad y = \lambda^2/H_1^2, \tag{7.1}$$

the governing differential equation becomes

$$\frac{dy}{dx} = \frac{y\{[\gamma(\gamma-1)x^2-2\gamma x+2]-y(1-x)^2[\gamma(\gamma+1)x^2-2(\gamma+1)x+2]\}}{x(\gamma x-1)\{y(1-x)^2(\gamma x-1)-(\gamma-1)x\}}. \tag{7.2}$$

Singularities of this equation occur at

$$\left.\begin{array}{ll}(a)\ x = 1/\gamma, y = 0; & (c)\ x = 0, y = 0;\\ (b)\ x = 1/\gamma, y = \gamma^2/(\gamma-1)^2; & (d)\ x = 0, y = 1.\end{array}\right\} \tag{7.3}$$

Of these (c) is not a simple singularity; the others have the following characters:

(a) saddle, (b) singular node, (d) saddle.

Further singularities may occur at points which satisfy the equations

$$\gamma(\gamma-1)x^2-2\gamma x+2-y(1-x)^2[\gamma(\gamma+1)x^2-2(\gamma+1)x+2] = 0, \tag{7.4}$$

$$(\gamma-1)x-y(1-x)^2(\gamma x-1) = 0, \tag{7.5}$$

of which, as usual, the former corresponds to the magneto-acoustic locus. Elimination of y between these two equations leads to the cubic equation

$$\gamma(\gamma-1)x^3+(\gamma^2-\gamma+2)x^2-2(\gamma+1)x+2 = 0,$$

which, for $\gamma = \frac{5}{3}$, has one real root at $x = -4{\cdot}08$. The singularity (e) associated with this root has the character of a spiral.

The behaviour of the integral curves in the neighbourhood of the singularity (c) may be determined from a consideration of the approximate form of (7.2) in the neighbourhood of this singularity, viz.

$$\frac{dy}{dx} = \frac{2y}{x\{y+(\gamma-1)x\}}.$$

This differential equation has the solution

$$\exp\left(\tfrac{1}{2}y\right)+\tfrac{1}{2}(\gamma-1)x\left[\log y+\sum_{n=1}^{\infty}\frac{y^n}{n\,2^n\,n!}\right] = kx \quad (y > 0),$$

where k is a constant. Therefore, for x positive

$$y \sim \exp(-1/x),$$

so that all integral curves to the right of the singularity enter the origin where they are tangential to the x axis. Similarly, for x negative

$$y \sim -1/|x|$$

and the nature of the singularity is saddle-like. The pattern of the full set of integral curves in the region of physical interest is shown in figure 3.

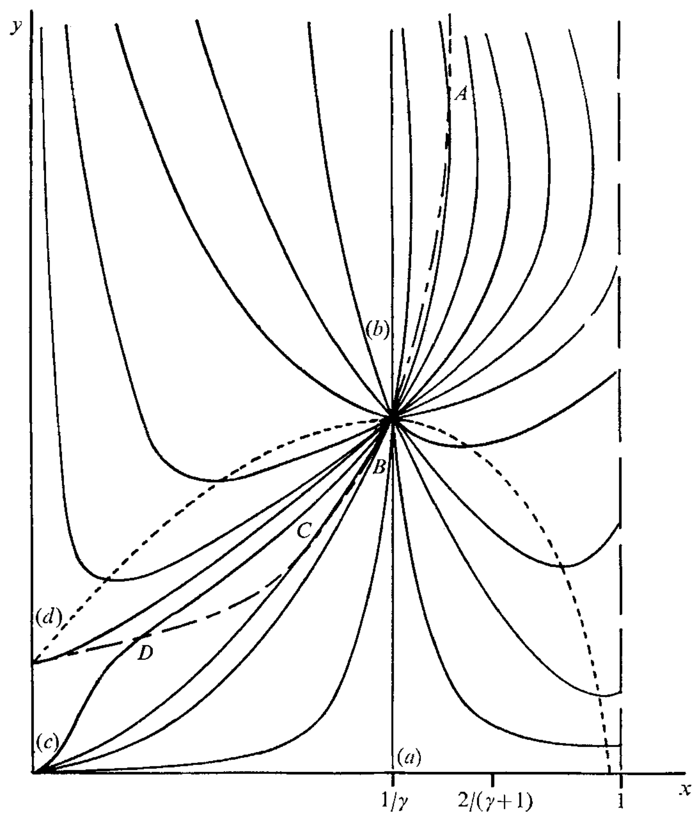

FIGURE 3. Magnetogasdynamic shock wave : integral curves.
– – –, acoustic locus;—– – —, shock locus.

Now in this instance, the value of V immediately behind the shock wave is derived from (4.1) as the smaller root of

$$(\gamma+1)V^2-(3+\gamma-\gamma H_1^2)V+2(1-H_1^2) = 0. \tag{7.6}$$

Thus by eliminating H_1^2 between (7.1) and (7.6) with $\lambda = 1$, one finds that the initial point on a solution curve lies upon the curve

$$y = \frac{(2-\gamma x)}{(1-x)[2-(\gamma+1)x]}, \tag{7.7}$$

which passes through the singularities (*b*) and (*d*) and has a vertical asymptote at $x = 2/(\gamma+1)$. This curve, the shock locus, is shown on figure 3 and for $x > 1/\gamma$ lies above and for $x < 1/\gamma$ lies below, the unique integral curve with equation

$$y(1-x)^2 = 1, \tag{7.8}$$

which also passes through the singularities (*b*) and (*d*). Thus to every value of x in the range $0 < x < 2/(\gamma+1)$ there corresponds an acceptable initial point from which a solution curve passes into the singularity (*c*) at the origin. Consequently in no instance does there arise a piston-driven shock wave.

In a detailed flow pattern the dimensionless field variables immediately behind the shock wave are obtained from (4.8)–(4.10) where, for given H_1, $V_2 = V(H_1)$ is the appropriate root of (7.6). The dimensionless field variables at any point on a solution curve are obtained from the integrals (3.1), (3.2) and (3.3), and are given as functions of (x, y) by

$$H = \frac{H_1}{1-x}, \quad P = \frac{(\gamma-1)H_1^2 x^2[y(1-x)^2-1]}{2(\gamma x-1)(1-x)^2}, \quad R = \frac{H_1^2 y}{1-x}. \tag{7.9}$$

The quantities of particular interest are the ratios of the dimensional field variables to their corresponding values immediately behind the shock wave, which, from the definitions (2.15) one may write in the forms

$$\frac{v}{v^*} = H_1\sqrt{y}\left[\frac{V}{V_2}\right], \quad \frac{p}{p^*} = \frac{1}{H_1^2 y}\left[\frac{P}{P_2}\right], \quad \frac{\rho}{\rho^*} = \frac{1}{H_1^4 y^2}\left[\frac{R}{R_2}\right], \quad \frac{\mathscr{H}}{\mathscr{H}^*} = \frac{1}{H_1\sqrt{y}}\left[\frac{H}{H_2}\right]. \tag{7.10}$$

The temperature and total pressure ratios in the gas may also be written

$$\frac{T}{T^*} = \frac{p/p^*}{\rho/\rho^*}, \quad \frac{\mathscr{P}}{\mathscr{P}^*} = \frac{P_2(p/p^*)+\frac{1}{2}H_2^2(\mathscr{H}/\mathscr{H}^*)^2}{P_2+\frac{1}{2}H_2^2}. \tag{7.11}$$

Since

$$H_1\sqrt{y} = \lambda = r/r^*, \tag{7.12}$$

these ratios may be conveniently obtained as functions of (r/r^*), viz. the proportion of the distance from the axial current to the shock wave at any instant.

For assigned H_1, if from (7.6) the corresponding value of V_2 is such that $V_2 < 1/\gamma$ then the complete solution may be obtained by following the appropriate integral curve into the singularity (*c*). However, if $V_2 > 1/\gamma$, this integral curve en route for (*c*) necessarily passes through the singularity (*b*). In this case, to avoid the computational difficulty which arises in integrating through the singularity, the value of H_1 is no longer specified and in its place the slope, m, of the integral curve at (*b*) is assigned. It is recalled that this value of m is bounded by the gradients at $x = 1/\gamma$ of the shock locus (7.7) and the curve (7.8) so that

$$2 < \frac{(\gamma-1)^3 m}{\gamma^3} \leqslant 3. \tag{7.13}$$

Then from an integration in each direction from (*b*) the corresponding value of H_1 and the complete solution may be derived. In the special instance when the initial point upon the shock locus itself coincides with the singularity (*b*) from (7.1) and (7.3) the value of $H_1 = (\gamma-1)/\gamma$ is known and hence, since from (4.10) one has $P_2 = 1/2\gamma^2$, it follows from the undetermined form (7.9) that the appropriate value of the slope m is given by the upper bound of the inequality (7.13). Detailed computations have been carried out when $\gamma = \frac{5}{3}$ for various values of H_1. Presented in figures 4(*a*)–(*f*) are the flow patterns associated with the following:

$$(A)\ H_1 = 0{\cdot}2, \quad (B)\ H_1 = 0{\cdot}4, \quad (C)\ H_1 = 0{\cdot}6, \quad (D)\ H_1 = 0{\cdot}8.$$

The shock point for each case is indicated upon figure 3 where it is observed that for (B) it coincides with the singularity (b).

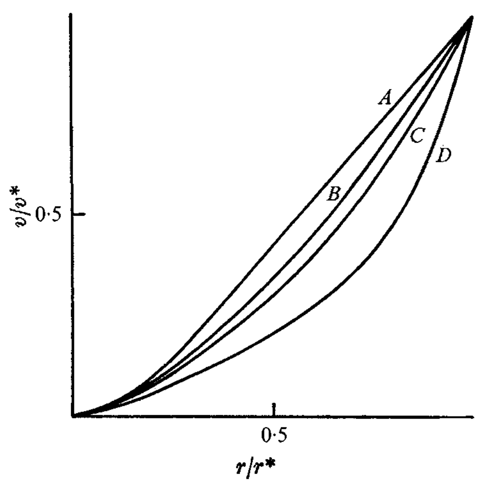

FIGURE 4(*a*). Magnetogasdynamic shock wave: velocity distribution. A, $H_1 = 0{\cdot}2$; B, $H_1 = 0{\cdot}4$; C, $H_1 = 0{\cdot}6$; D, $H_1 = 0{\cdot}8$.

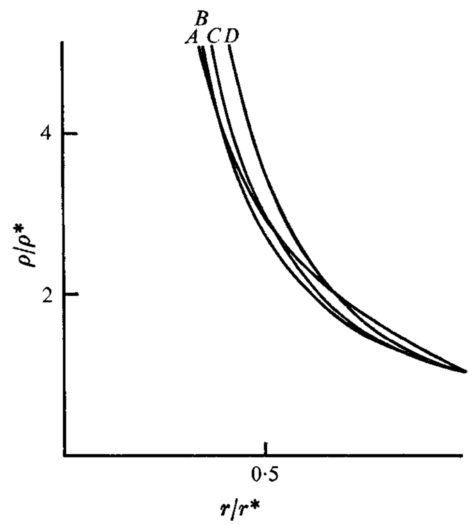

FIGURE 4(*b*). Magnetogasdynamic shock wave: density distribution.

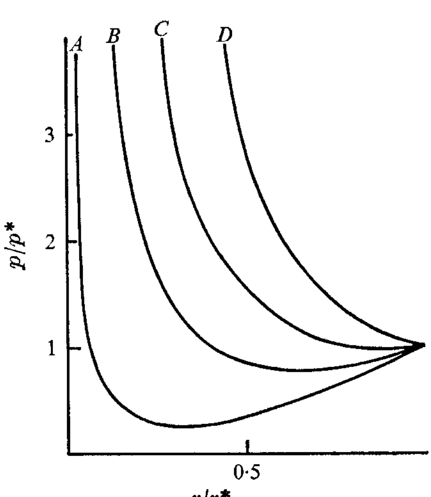

FIGURE 4(*c*). Magnetogasdynamic shock wave: pressure distribution.

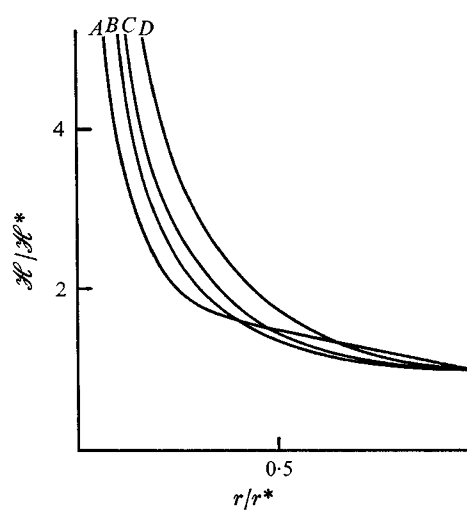

FIGURE 4(*d*). Magnetogasdynamic shock wave: magnetic field distribution.

It is noted that, for the smaller values of H_1, the pressure falls behind the shock front before finally rising to an infinite value at the axis of symmetry, whilst at any particular station, the pressure ratio increases with the value of H_1. Furthermore, as $H_1 \to 0$, the shock point has co-ordinates $x \to 2/(\gamma+1)$, $y \to \infty$ and the appropriate integral curve tends to become coincident with the limiting curve given by (7.8). Thus, since the pressure is zero upon the latter, it follows that as $H_1 \to 0$, the pressure decreases to zero everywhere behind the shock wave.

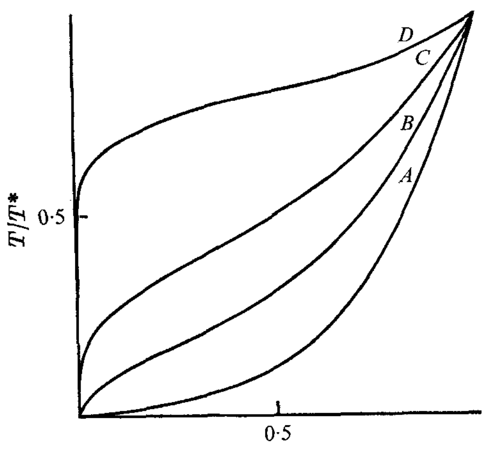

FIGURE 4 (*e*). Magnetogasdynamic shock wave : temperature distribution.

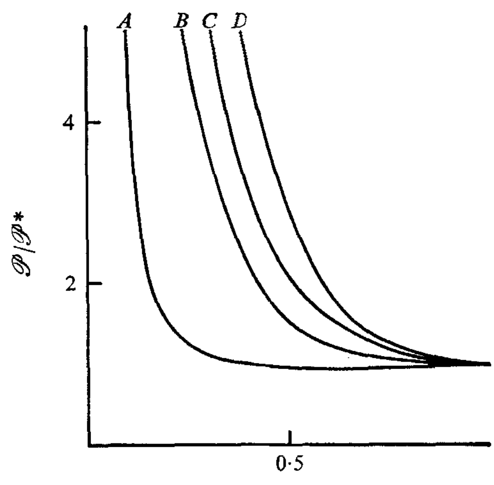

FIGURE 4 (*f*). Magnetogasdynamic shock wave : total pressure distribution.

As may be expected from the frozen flux relationship the curves of density and magnetic field variation have similar properties. In fact, from (7.10)–(7.12) t follows that $\mathscr{H}/\mathscr{H}^* = (r/r^*)(\rho/\rho^*)$. For larger values of H_1 at any fixed r/r^*, both ratios increase with H_1 in the same manner as the pressure, but for smaller H_1 it is observed that this trend is reversed in the region nearer the shock front and only becomes common for all H_1 sufficiently far behind the shock wave. An associated effect is shown by the velocity ratio, the reduction in which, at any station, increases with the increase of H_1.

It is observed further that, at the axis of symmetry where there flows the line current the velocity falls to zero whilst the pressure, density, magnetic field and thus total pressure all rise to infinitely large values. The solution thus cannot be valid in the immediate vicinity of the axis, a fact which is emphasized by the result that there also the temperature falls sharply to zero, a detail which is hardly consistent with the fundamental assumption that the gas remains perfectly conducting throughout.

Some of the above effects may be clarified by a consideration of the electric field, **E**, and the current density, **j**. These are given by the appropriate Maxwell's equation and may be written in the forms

$$\mathbf{E} = \frac{-\mu x H_1}{t(1-x)}\mathbf{z}, \quad \mathbf{j} = \frac{2y}{r^2\mu^{\frac{1}{2}}}\frac{Q}{(1-x)^2}\frac{dx}{dy}\mathbf{z}, \tag{7.14}$$

where $\mathbf{z}$ is a unit vector parallel to the line of the axial current. Thus, the electric field, at any time subsequent to the initial instant of propagation, is finite, varying from zero at the axis of symmetry to $-\mu V_2 H_1/t(1-V_2)$ at the wave front. Also if J is the total current flowing in the gas in a direction parallel to that of the central current then, following some algebraic reduction,

$$J = \int_0^{ct} 2\pi r j\, dr = \frac{IV_2}{1-V_2},$$

which is finite and positive. However, from (7.14) we see that the sign of the current density depends upon the sign of dy/dx. Flow patterns corresponding to large H_1 with shock points such as D, C, B (see figure 3) are characterized by the fact that dy/dx is always positive, whilst those with small H_1 and shock points such as A have dy/dx negative near the initial point on the solution curve but positive subsequently. Thus for small H_1 the local current flow is parallel to the applied current near the axis, but antiparallel near the wave front. The value of $H_1 = \bar{H}$ for which the current density does not change sign is given by $\bar{H} = y^{*\frac{1}{2}}$ where y^* is the ordinate of the point of intersection of the shock locus (7.7) and the locus of infinite slopes upon integral curves, viz.

$$y = \frac{(\gamma-1)x}{(1-x)^2(\gamma x-1)}.$$

In particular if $H_1 = \bar{H}$ then the current density falls to zero at the wave front.

Finally, the Lorentz force acting upon the gas particles may be computed. For $H_1 > \bar{H}$ the current density is one signed and the Lorentz force thus acts towards the axis and served to dampen the flow. Alternatively if $H_1 < \bar{H}$, immediately behind the wave front the Lorentz force acts away from the axis and the motion is assisted, whilst near the axis the effects are reversed. Thus if we examine the solutions corresponding to cases B, C and D, for which $H_1 > \bar{H}$ we note that the gas in these instances is subject to a force opposing any motion and increasing with H_1. This has the effect of restraining a greater mass of gas in the neighbourhood of the axis of symmetry, of reducing the gas velocity and increasing the total pressure with H_1. In the case A the Lorentz force changes sense and thus is observed to reduce the total pressure in the gas as one moves in behind the wave front, but then as the sense changes and the force increases beyond bound, to boost the pressure to an infinite value at the axis. In this instance there arises a tendency to segregate the gas into two regions, one near the wave front, the other near the axis in which the rate of increase of the density is less at the axis and greater at the wave front compared with the situation when $H_1 > \bar{H}$.

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